

CHAPTER 3

DETERMINANTS.

Reconsider the computation of the inverse of

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

"A is invertible iff $ad - bc \neq 0$ "

The quantity $ad - bc$ is called $\det(A)$ and it's also denoted as $|A|$. It means

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Minors and Cofactors

To extend the concept of determinant to higher order square matrices, it's convenient to define $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

And then

$$\det(A) \stackrel{\text{def}}{=} a_{11}a_{22} - a_{12}a_{21} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Definition of $\det(A)$ for a 3×3 matrix from the definition of $\det(A)$ for a 2×2 matrix.

Consider $A = \begin{bmatrix} 1 & 4 & -3 \\ -2 & -7 & 6 \\ 1 & 7 & -2 \end{bmatrix}$

1st row $A = \begin{bmatrix} 1 & 4 & -3 \\ -2 & -7 & 6 \\ 1 & 7 & -2 \end{bmatrix}$

$$M_{11} = \begin{vmatrix} -7 & 6 \\ 7 & -2 \end{vmatrix} = 14 - 42 = -28.$$

$$A = \begin{bmatrix} 1 & 4 & -3 \\ -2 & -7 & 6 \\ 1 & 7 & -2 \end{bmatrix}, \quad M_{12} = \begin{vmatrix} -2 & 6 \\ 1 & -2 \end{vmatrix} = 4 - 6 = -2$$

$$A = \begin{bmatrix} 1 & 4 & -3 \\ -2 & -7 & 6 \\ 1 & 7 & -2 \end{bmatrix}, \quad M_{13} = \begin{vmatrix} -2 & -7 \\ 1 & 7 \end{vmatrix} = -14 + 7 = -7$$

Observe

$$a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} = 1(-28) - 4(-2) - 3(-7) = -28 + 8 + 21 = 1.$$

Definitions:

a) M_{11}, M_{12}, M_{13} are called "minors"

M_{11} is the minor of entry a_{11} and so forth...

b) We also define

$$C_{11} = (-1)^{1+1} M_{11} = M_{11}, \quad C_{12} = (-1)^{1+2} M_{12} = -M_{12}, \text{ and so forth.}$$

$$\text{Then, } a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} = 1$$

Also, 1st Column

(2')

$$A = \begin{bmatrix} 1 & 4 & -3 \\ -2 & -7 & 6 \\ 1 & 7 & -2 \end{bmatrix}, M_{21} = \begin{vmatrix} 4 & -3 \\ 7 & -2 \end{vmatrix} = 13$$

$$A = \begin{bmatrix} 1 & 4 & -3 \\ -2 & -7 & 6 \\ 1 & 7 & -2 \end{bmatrix}, M_{31} = \begin{vmatrix} 4 & -3 \\ -7 & 6 \end{vmatrix} = 3$$

and

$$a_{11}M_{11} - a_{21}M_{21} + a_{31}M_{31} = 1(-28) - (-2)(13) + 1(3) = 1$$

$$\text{Also, } M_{22} = \begin{vmatrix} 1 & -3 \\ 1 & -2 \end{vmatrix} = -2 + 3 = 1$$

$$M_{23} = \begin{vmatrix} 1 & 4 \\ 1 & 7 \end{vmatrix} = 7 - 4 = 3$$

$$M_{32} = \begin{vmatrix} 1 & -3 \\ -2 & 6 \end{vmatrix} = 6 - 6 = 0$$

$$M_{33} = \begin{vmatrix} 1 & 4 \\ -2 & -7 \end{vmatrix} = -7 + 8 = 1$$

$$\text{Also, } a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} = 1$$

where $C_{i1} = (-1)^{i+1} M_{i1}, i=1,2,3.$

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The Common number "1" obtained independently of the row or column considered is called $\det(A)$.

Therefore,

$$\boxed{\det(A) = a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13} = 1} \quad (2.1)$$

↗ Along row 1

The numbers c_{11}, c_{12}, c_{13} are called "Cofactors" of their respective entries a_{11}, a_{12}, a_{13} .

But also,

$$\boxed{\det(A) = a_{11}c_{11} + a_{21}c_{21} + a_{31}c_{31} = 1} \quad (2.2)$$

↗ Along column 1.

The expression (2.1) is called cofactor expansion of A along 1st row.

and (2.2) correspond to a cofactor expansion of A along 1st column.

Also, 2nd Row

$$-a_{21} M_{21} + a_{22} M_{22} - a_{23} M_{23} =$$

$$= 2(13) + (-7)(1) - 6(3) = 26 - 7 - 18 = 1.$$

3rd Row

$$a_{31} M_{31} - a_{32} M_{32} + a_{33} M_{33} =$$

$$= 1(3) - (-7)(0) + (-2)(1) = 1$$

Similarly, the corresponding combination for columns

Column 2

$$-a_{12} M_{12} + a_{22} M_{22} - a_{32} M_{32} = 1$$

Column 3

$$a_{13} M_{13} - a_{23} M_{23} + a_{33} M_{33} = 1$$

This common number "1" will be called the

$$\det(A), \text{ i.e., } \underline{\det(A) = 1}$$

Definition. - the determinants M_{ij} ($i, j = 1, 2, 3$) of the 2×2 matrices that remain after the i th row and j th column are deleted from $A_{3 \times 3}$ are called the minors of entry a_{ij} . The number $C_{ij} \equiv (-1)^{i+j} M_{ij}$ is called the cofactor of entry a_{ij} .

The property observed in the previous computation for this particular 3×3 matrix is true in general for any 3×3 matrix A .

This is the content of the following theorem:

Thm. - If A is a 3×3 matrix, then regardless of which row or column of A is chosen, the number obtained by multiplying the entries in that row or column by the corresponding cofactors and adding the resulting products is always the same.

This property can be easily proved for any 2×2 matrix. In fact,

for $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$M_{11} = a_{22}, \quad M_{12} = a_{21}, \quad M_{21} = a_{12}, \quad M_{22} = a_{11}$$

$$C_{11} = a_{22}, \quad C_{12} = -a_{21} = -M_{12}, \quad C_{21} = -a_{12} = -M_{21}, \quad C_{22} = a_{11}$$

and

$$a_{11}C_{11} + a_{12}C_{12} = a_{11}a_{22} - a_{12}a_{21} \quad (\text{along 1st row})$$

$$a_{11}C_{11} + a_{21}C_{21} = a_{11}a_{22} - a_{21}a_{12} \quad (\text{along 1st column}).$$

Similar for 2nd row and 2nd column.

Therefore, the previous thm is also verified by 2×2 matrices.

Def. - If A is a 3×3 matrix, then the common number obtained by multiplying the entries in any row or column of A by the corresponding cofactors and adding the resulting products is called the determinant of A , and the sum themselves are called cofactor expansions of A . That is,

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + a_{3j}C_{3j}, \text{ for any column } j=1,2,3$$

(Cofactor expansion along j th column)

And

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3}, \text{ for any row } i=1,2,3$$

(Cofactor expansion along i th row)

Definition of $\det(A)$ for an $n \times n$ matrix

the previous definition ^{about minors and cofactors} can be easily extended to 4×4 and inductively to any $n \times n$ matrix. Also, the theorem remains true for the $n \times n$ matrices. Therefore

the definition of $\det(A)$ for an $n \times n$ matrix is a straightforward extension of the one for 3×3 matrices.

Given above. More specifically,

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \dots + a_{nj}C_{nj}, \text{ for any column } j=1,2,\dots,n$$

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in}, \text{ for any row } i=1,2,\dots,n$$

Computing $\det(A)$ by a smart choice of Row or Column.

2.1) #21)

$$A = \begin{bmatrix} -3 & 0 & 7 \\ 2 & 5 & 1 \\ -1 & 0 & 5 \end{bmatrix}$$

Choose 2nd column, then

$$\begin{aligned} \det(A) &= 0 C_{12} + 5 C_{22} + 0 C_{32} = \\ &= 5 M_{22} = 5 \begin{vmatrix} -3 & 7 \\ -1 & 5 \end{vmatrix} = \\ &= 5 (-15 + 7) = -40. \end{aligned}$$

Arrow technique for 3×3 matrices

11)

$$A = \begin{bmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{bmatrix} \begin{matrix} -2 & 1 \\ 3 & 5 \\ 1 & 6 \end{matrix} = (-20 - 7 + 72) - (20 + 84 + 6) = 45 - 110 = -65$$

$$\frac{72}{27} \\ 45$$

Same result that will be obtained by applying Cofactor expansion. See proof in book.

Determinant of an Upper Triangular matrix.

$$\det \begin{pmatrix} \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \end{pmatrix} = a_{11} \begin{vmatrix} a_{22} & 0 \\ a_{32} & a_{33} \end{vmatrix} = a_{11} a_{22} a_{33}$$

Inductively can be proven that $\begin{vmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \dots & \dots & a_{nn} \end{vmatrix} = a_{11} a_{22} \dots a_{nn}.$

Consider

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 4 & 5 & 6 \end{bmatrix}$$

Then,

$$\det A = 1 \begin{vmatrix} 2 & 0 \\ 5 & 6 \end{vmatrix} - 0 \begin{vmatrix} 3 & 0 \\ 4 & 6 \end{vmatrix} + 0 \begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix} = 1 \times 2 \times 6 = 12.$$

This result is true for any matrix $A_{n \times n}$.

Thm. 2. - If

$$A = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \vdots & & \ddots & \vdots \\ a_{n1} & \dots & \dots & a_{nn} \end{bmatrix} \text{ is a "triangular matrix"}$$

then,

$$\det A = a_{11} a_{22} \dots a_{nn}.$$